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altogether satisfactory have been either suppressed or altered; and information which had become superfluous by reason of change in the law or general opinion, has been carefully eliminated; while the greatest care has been taken to introduce, wherever requisite, all the necessary information consequent upon the changes which have taken place in life assurance itself, the later decisions in the Courts of Justice and the subsequent alterations in the Law. But in some minor points further revision might with advantage have been bestowed on the work. Thus in the foot note on page 4, the Gambling Act is cited as 14 Geo. III., c. 38, whereas it really is c. 48. This is evidently a printer's error, as is, no doubt, a similar error upon p. 390, where the case of Fenn *v.* Edmonds is cited as Fenn *v.* Edwards. Again, the word "late," as applied to cases cited, has been retained too frequently in this edition; and on p. 219 a case is still termed "very recent," which was so termed in the first edition. Surely 14 or 15 years will reduce a case from the class of "very recent" to a lower grade. Altogether this work forms a most complete treatise on the Law of Life Assurance, and will be found a most valuable addition to both the legal and assurance libraries; and it will no doubt obtain, as it certainly deserves, an extensive sale both among lawyers and actuaries.

INSTITUTE OF ACTUARIES.

SOLUTIONS OF THE SECOND YEAR'S EXAMINATION QUESTIONS, 1869; WITH REMARKS. BY THE EXAMINERS, PETER GRAY, F.R.A.S., AND RALPH P. HARDY.

1. Define the logarithm of a number.

Ans. The logarithm of a number is the index of that power of the base of the system in which the logarithm is taken, which is equal to the number.

Thus, in the equation

$$b^x = a,$$

x is the logarithm of a in the system whose base is b.

The object in view in proposing this question was to test the possession on the part of the candidates of the power of clear conception and correct definition, a power in constant requisition in dealing with questions involving complex contingencies. We find not unfrequently the logarithm of a number stated to be "that power of the base which is equal to the number." A power is here confounded with its index. In the above equation b^x is *the power*,—the xth power—of b; and it is *equal* to a. It is the index, x, which is the logarithm of a.

This very loose mode of expression, if it did not originate with the late D. Jones, at least received abundant countenance from him. In his work on *Annuities* we are frequently directed to raise a certain quantity to "the same power as the number of years," his meaning being, to "the power whose index is the number of years." See his vol. i., pp. 11, 20, 27, &c.

2. The system of logarithms known as Briggs's, or the common system, is more convenient for numerical computations than any other. State the reason.

Ans. We have two advantages in the use of Briggs's system of logarithms:—First, that of two or more numbers consisting of the same succession of digits, but differing in the position of the decimal point, the mantissæ of the logarithms are the same; and secondly, as incidental to the first, that by inspection of a number we are able at once to assign the characteristic, or integer portion, of its logarithm. And these advantages arise in consequence of the radix of our numeral system and the base of Briggs's system of logarithms being identical.

Let a be any number. Then, if r be the radix of the numeral system, and n any integer, a and ar^n will consist of the same succession of digits, and the point of separation between the integer and the fractional part of ar^n will be n places to the right or the left of its position in a, according as n is positive or negative.

Taking the logarithms *in any system* we should have

$$\log ar^n = \log a + n \log r.$$

But in every system the logarithm of its base is unity. Hence if we use the system whose base is r, the equation will become

$$\log ar^n = \log a + n.$$

In this system then, which is Briggs's, the logarithms of all numbers consisting of the same succession of digits, differ only in the value of the integer n, which thus indicates the number of places by which the position of the separating point differs in the numbers.

Again. In a numeral system whose radix is r the nth power of r (or r^n), n being an integer, is an integer consisting of unity followed by n ciphers; and in like manner r^{n+1} is an integer consisting of unity followed by $n+1$ ciphers. Now by definition n and $n+1$ are the logarithms of r^n and r^{n+1} to base r, respectively. Hence the logarithms of all numbers comprised between r^n and r^{n+1}, that is of all numbers whose integer part consists of $n+1$ digits, will be comprised between n and $n+1$. Their characteristic or integer portion will therefore be n.

In virtue of the possession of this property it suffices to tabulate only the mantissæ of Briggs's logarithms.

3. An homogeneous coin is tossed three times. Assign the probability of the result giving, (a) one head; (b) two heads; (c) three heads. The sum of these probabilities differs from unity. Why?

Ans. Such a very simple question as this admits of solution on the most purely elementary principles. Write down all the results that can arise from three tossings, thus:—

H H H
H H T
H T H
H T T
T T T
T T H
T H T
T H H

They are eight in all; and they are all equally probable, since in a single trial the probability of H is the same as that of T. Of these eight results, three (Nos. 4, 6, 7) favour (a); three (Nos. 2, 3, 8) favour (b); and one (No. 1) favours (c). Hence the probabilities of these several events are $\frac{3}{8}$. $\frac{3}{8}$, and $\frac{1}{8}$, respectively.

But the solution of all such problems is best effected by the aid of the expansion of the binomial $p+q$, p and q denoting respectively the probabilities of the two contrary events (here the occurrence of head and the occurrence of tail) in a single trial. Raising to the *third* power, there being here *three* trials,

$$p^3+3p^2q+3pq^2+q^3,$$

we have in each term the probability of the occurrence indicated by the argument of that term. Thus, p and q being each $\frac{1}{2}$, we have from the expansion the probabilities of the various results, as follows:—

1st term, $\frac{1}{8}$, 3 heads.
2nd " $\frac{3}{8}$, 2 heads and 1 tail, in any order.
3rd " $\frac{3}{8}$, 1 head and 2 tails, " "
4th " $\frac{1}{8}$, 3 tails.

The first three terms, in reverse order, give the probabilities required.

The events whose probabilities are p and q being contrary events, $p+q$ is equal to unity. Hence also $(p+q)^n=1$, for all values of n; in other words, the sum of the terms of the expansion of $(p+q)^n$, is always unity. The meaning of this is, that, the terms of the expansion being the probabilities of *all* the compound events which can arise in n trials, their sum is the probability that *one or other of them* will arise; and this is an event of which, as it is certain to happen on the n trials being made, the probability is unity.

From the above it follows, that if the sum of the probabilities, correctly determined, of a number of the events that can arise out of n trials, does not amount to unity, one or more of the possible events must have been omitted. In the present case the omission is, the occurrence of three tails.

Galloway *On Probability* may be advantageously studied.

4. A coin being tossed as before, assign the probabilities of the following results: (*a*) one head at least; (*b*) two heads at least; (*c*) three heads at least.

Ans. The probabilities here required are the sums of the probabilities of the possible cases in which (*a*) one head or more occur, (*b*) two heads or more, (*c*) three heads. Thus we have,

$$\begin{array}{lll} (a) & p^3+3p^2q+3pq^2 & =\frac{1}{8}+\frac{3}{8}+\frac{3}{8}=\frac{7}{8}; \\ (b) & p^3+3p^2q & =\frac{1}{8}+\frac{3}{8} \quad =\frac{1}{2}; \\ (c) & p^3 & \qquad\qquad =\frac{1}{8}. \end{array}$$

The sum here differs from unity because the events in question are not *contrary* events.

5. Give the formulæ which represent the accumulated amount of an annuity, when the first payment is made in advance:—

(1) When the annuity is receivable annually;
(2) When the annuity is receivable half-yearly.

Ans. (1). The amount of an annuity for n years, at the period of the last payment being made is

$$\frac{(1+i)^n-1}{i}.$$

If paid in advance, each payment will have been improved for one year more than in the former case, and the amount at the same period will consequently be

$$\frac{\{(1+i)^n-1\}(1+i)}{i}=\frac{(1+i)^n-1}{1-v}.$$

(2). The forms of the foregoing expressions are in no degree due to the circumstance that the interval between successive payments is a year. They will hold in reference to any interval when i denotes the interest accruing on one pound during one such interval, and n denotes the number of payments. If then we retain n to denote the number of years in the duration of an annuity, and i the yearly interest on one pound, to adapt the above formulæ to half-yearly payments, we must write in them $2n$ for n, $\frac{1}{2}i$ for i, let v denote $1\div(1+\frac{1}{2}i)$, (writing it in this case v_1,) and finally multiply by $\frac{1}{2}$, each payment now being half one of the yearly payments. The formulæ then become

$$\frac{1}{2}\cdot\frac{(1+\frac{1}{2}i)^{2n}-1}{\frac{1}{2}i}, \text{ and } \frac{1}{2}\cdot\frac{(1+\frac{1}{2}i)^{2n}-1}{1-v_1}, \text{ respectively.}$$

The usual tables of the amounts of annuities, readily give the amounts when the annuities are payable in advance. Thus the amount of an annuity for n years, payable in advance, is the same as the tabular amount for $n+1$ years, diminished by £1, its last payment.

So also the common tables can be adapted to the case of annuities payable half-yearly, &c. Thus the amount of an annuity for n years, payable half-yearly at a specified rate, is half the tabular amount of an annuity for $2n$ years, at half the rate.

These properties are so obvious as not to stand in need of proof.

6. A and B (and their representatives) are put in possession, in equal shares, of an annuity certain of £a for $2n$ years. They arrange to take the payments alternately, A taking the first. How much ought he now to pay to B for the advantage he thus receives?

Ans. The method of proceeding which would here probably first suggest itself would be to take the difference between the sums of the following series, which express the values of the payments to be received by A and B, respectively, viz:—

$$a(v+v^3+v^5+\ \ldots\ +v^{2n-1},)$$

and

$$a(v^2+v^4+v^6+\ \ldots\ +v^{2n};)$$

but the trouble of summing the series may be avoided thus:—

We know the value of the entire annuity, which is $\frac{a(1-v^{2n})}{i}=\text{M}$, say; and we know also that the values of the payments receivable by A and B are in the ratio of $1:v$. Hence, if x denote the value of A's payments, that of B's will be denoted by vx. We shall consequently have

$$(1+v)x=\text{M}; \text{ whence } x=\frac{\text{M}}{1+v}.$$

$$\text{and}\quad vx=\frac{v\text{M}}{1+v}.$$

The difference of these is $\frac{(1-v)\text{M}}{1+v}$, which is the excess of the value of A's payments over that of B's; and the shares of the two parties will be equalised by A making payment to B of half this amount, namely $\frac{(1-v)\text{M}}{2(1+v)}$.

If we restore the value of M, the value just deduced becomes

$$\frac{av(1-v^{2n})}{2(1+v)}.$$

7. State the leading characteristics of the Carlisle Table of Mortality. State also for what class of risks you consider it to be best adapted.

Ans. The Carlisle Table of Mortality was constructed by the late Mr. Milne, his data being two enumerations of the inhabitants of Carlisle, having between them an interval of nearly eight years,* with the registered number of births and deaths during the same interval; all as contained in a tract published by Dr. Heysham. In both enumerations the ages are given for the same quinquennial periods. But it has been recently pointed out by Mr. Makeham (*Journal*, vol. xii., pp. 319-21) that in both enumerations the distribution of the numbers living amongst the several periods is absolutely identical. This seems to render it all but certain that in only one of them—the first—were the ages distinguished. If so, and as the numbers were but few, we have most likely in this circumstance an explanation of those irregularities in the indications of the Carlisle Table of Mortality, which must be held to impair to some extent its authority as an exponent of healthy life in England. As an illustration of what is here referred to it may be mentioned that this table gives the mean duration for ages 92 to 97 greater than that for age 91.

8. Explain De Moivre's hypothesis as to the probabilities of life; and state whether it indicates a stationary or a variable rate of mortality, with the reasons for your opinion.

Ans. De Moivre's hypothesis is that of 86 persons born one dies each year till all are extinct. It was not proposed by its author as giving a *correct* representation of the probabilities of life, but as affording the means of readily deriving the usual results of the mortality table, while the results so formed should not, at least for the middle ages of life, deviate much from the truth.

Take any age, x. Then $86-x=c$, is called the complement of life; and if the numbers living were arranged as in the mortality table, we should have $l_x=c$. But it is usual to express the tabular functions in terms of c. Thus, we have,

$$p_{x.n}=\frac{c-n}{c}$$

$$e_x=\frac{c}{2}$$

And if $a_{(c)}$ denote the present value of an annuity certain for c years, we further have:

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* This interval is usually stated to have been nine years. But January, 1780, to December, 1787, gives only seven years and eleven months.

$$a_x = \frac{1}{ci}\{c-(1+i)a_{(c)}\}$$

$$\mathrm{A}_x = \frac{a_{(c)}}{c}$$

$$\varpi_x = \frac{(1-v)a_{(c)}}{c-a_{(c)}}$$

This hypothesis indicates an *increasing* rate of mortality. The probability of dying in a year is a fraction whose numerator is constant, being unit, and whose denominator, c or l_x, decreases as x, the age, increases.

9. Give expressions for the following functions; and state clearly the meaning attached to them, and the practical uses to which they can be put:—(1) Expectation of life; (2) *Vie probable;* (3) Mean age at death.

Ans. (1) Expectation of life, *vide* Gray's "Tables and Formulæ," § 135, *et seq.* See also *Journal*, vol. xiii., p. 381.

(2) *Vie probable, vide* Gray's "Tables and Formulæ," § 170.

(3) Mean age at death, *vide* Farr's Introduction to English Life Table No. 3, p. xxxv.

No essentially practical use has yet been made of these functions. The values of some benefits might be computed from the "expectation of life"; but the process would be laborious. For an application, *vide Journal*, vol. iii., 325.

10. How do you express symbolically the probability that an assigned life will survive a year? Explain what is meant by such a function, and state under what conditions and limits it is a fair representation.

Ans. By $p_x = \frac{l_{x+1}}{l_x}$; by the ratio between the number attaining age x, and the number attaining age $x+1$. Assuming that the life selected be a fair specimen of its class, and that the mortality of the class be adequately represented by the table of mortality employed, this ratio is the probability, or a numerical expression of the force of the impression upon our minds before the determination of the event, certainty being represented by unity.

11. State the constitution and characteristic properties of the Single Life Commutation Table, and show the relations which subsist between corresponding values in the several columns.

Ans. See *Journal of the Institute*, vols. xii., p. 328, and xiii., p. 129.

12. If ϖ_x and ϖ_{x+1} represent the annual premiums for the ordinary whole life assurance at ages x and $x+1$ respectively, show symbolically the probability that (x) will die in a year:—

(1) If the premiums be taken as net;

(2) If the premiums be taken as loaded to the extent of 25 per cent.

Ans. We have $$\varpi_x = \frac{1}{1+a_x} - (1-v);$$

$\therefore$ $$\frac{1}{1+a_x} = \varpi_x + (1-v),$$

and $$1+a_x = \frac{1}{\varpi_x + 1 - v} \quad . \; . \; . \; . \; . \; (1)$$

Similarly, $$1+a_{x+1}=\frac{1}{\varpi_{x+1}+1-v}\quad\ldots\ldots\quad(2)$$

Now $$a_x=vp_x(1+a_{x+1})=\frac{vp_x}{\varpi_{x+1}+1-v},$$ from (2) by substitution.

But from (1), $$a_x=\frac{v-\varpi_x}{\varpi_x+1-v}.$$

Hence, equating, $$\frac{vp_x}{\varpi_{x+1}+1-v}=\frac{v-\varpi_x}{\varpi_x+1-v};$$

whence, $$p_x=\frac{v-\varpi_x}{v}\cdot\frac{\varpi_{x+1}+1-v}{\varpi_x+1-v}\quad\ldots\ldots\quad(3)$$

And $$1-p_x=\frac{\varpi_x-(v-\varpi_x)\varpi_{x+1}}{v(\varpi_x+1-v)},$$

which is the probability of dying in a year.

The use of this problem is to enable us, when a table of premiums is given, to deduce thence the mortality on which it is founded. To do this with certainty it is obviously necessary to know the rate of interest involved, and also the loading, if any, that the premiums bear. If these be known or assumed, $\log p_x$ for each age can easily be determined by (3).

If the loading be p ꝑ £, the premiums must be diminished in the ratio of $1+p$ to 1 before being used in the formula.

It appears from (1) and (2) that (3) may be written as follows :

$$p_x=\frac{v-\varpi_x}{v}\cdot\frac{1+a_x}{1+a_{x+1}}.$$

Papers on this subject will be found in the *Assurance Magazine*, vol. ii., p. 391, by Mr. William Wylie; vol. vi., p. 231, by Mr. George Scott; and vol. vi., p. 297, by H. A. S.

13. If the claims expected by an Office in a given year were more than those actually experienced, can the amount stated to have been so saved be termed profit? Give full reasons for your opinion.

Ans. No: the whole of this difference would not appear in the surplus upon a valuation of the assets and liabilities of the Office. The liability is not extinguished by the non-occurrence of the exact number of expected deaths within the specified period. The mortality in respect of these suspended claims may emerge gradually or at once;—may be distributed over future years according to the normal rate, or follow closely upon the period of rest.

In the first case, the profit would be the difference in question less the reserve necessary to be held against such suspended cases. In the second, merely the interest upon the sum assured so long as it is not called for, and the accumulation of premiums received in the meanwhile.

Let the reserve fund of the Office be $l_{40}V_{30|10}$, and let the deaths of the ensuing year be 60, instead of 66 as provided by the Carlisle Table.

Then, the difference between the expected and the accrued claims will be 6. How much of this may be appropriated as profit?

Case 1. If the mortality of the six unexpected survivors (*i.e.* 5009+6) be assumed to follow the normal rate, the profit

$$\begin{aligned} P &= 6(1-V_{30|11}) \\ &= 5{\cdot}221 \text{ at Car. 3 per cent.} \end{aligned}$$

This is the greatest amount of profit resulting from the year's favourable experience, and would be properly appropriated if selection has a calculable value.

Case 2. If these six deaths be only postponed, and if the time of postponement be assignable—say n years—so that the mortality of the nth year would be excessive in respect of these deaths, then the sums suspended might be carried to a special account, and the present value of both the interest on the same and the accumulation of the premiums meantime receivable, might be appropriated as profit. Hence

$$P=v^n\{6((1+i)^n-1+\varpi_x(A_{(n+1)}-1))\}$$

Taking $n=1$, we get $P=6\{\varpi_x+(1-v)\}$.

$$\begin{aligned} P &= {\cdot}9708\times 6({\cdot}03+{\cdot}01952\times 1{\cdot}03) \\ &= {\cdot}2919 \text{ at Car. 3 per cent.} \end{aligned}$$

14. Find the formula for the present value of the remainder of an annuity certain for n years, to be entered upon at the death of (x), the present possessor.

Ans. The entire value of the annuity is $a_{(n)}$, and the value of (x)'s share of it is ${}_{\overline{n}|}a_x$. Hence the value of what remains is

$$a_{(n)}-{}_{\overline{n}|}a_x.$$

This expression cannot become negative, since ${}_{\overline{n}|}a_x$ is always less than $a_{(n)}$. For $a_{(n)}$ is $v+v^2+\ldots+v^n$ (n terms), while ${}_{\overline{n}|}a$ is $vp_x+v^2p_{x.2}+\ldots+v^np_{x.n}$ (also n terms), and the co-efficients, p_x, &c., are all less than unity.

15. Investigate a formula for the annual premium for an assurance on the life of (x), which is to be 1,2,3, . . . n pounds according as the death of (x) shall take place in the 1st, 2nd, 3rd . . . nth year, and to remain constant at the last-named amount during the remainder of life:—

(1) For a premium payable n times, provided (x) so long lives;
(2) For a premium payable throughout life.

Ans. Here the benefit side of the equation is,

$$R_x-R_{x+n};$$

and the payment side, according as the premium is payable for n years or for life, is

$$\varpi(N_{x-1}-N_{x+n-1}), \text{ or } \varpi_1 N_{x-1}.$$

Equating, we get

$$\varpi=\frac{R_x-R_{x+n}}{N_{x-1}-N_{x+n-1}}, \text{ and } \varpi_1=\frac{R_x-R_{x+n}}{N_{x-1}}.$$

16. Deduce the expression for the single premium for an assurance on (x) against (y).

Ans. The sum assured, say one pound, will be receivable in the nth year if (x) die therein, and if he be survived by (y). There is no restriction as to the duration of the term of (y)'s survivorship: the second condition will be fulfilled if he be alive *at the instant* of (x)'s death.

It is at once obvious that for the solution of this problem it is necessary to take account of the manner in which the deaths of the nth year, in each of the classes to which (x) and (y) respectively belong, are distributed over the year. It is usual to assume that the deaths in each class are uniformly distributed over the year in which they take place; so that the deaths which occur amongst that class in any specified portion of the year are the same in number as those that occur in any other equal portion. This assumption, which would be rigorously true if the annual decrements of the mortality table were equal for the whole duration of life, is, as matters stand, sufficiently near the truth for practical purposes. The decrements, except for a few years at the close of life, vary but slowly; and the error committed by assuming them to be uniformly distributed in each year, is found to be immaterial. In what follows the usual assumption is made.

Denote, in the meantime, for brevity, the number of (x)'s class now alive by A, of whom a enter on the nth year, and d die therein; and denote the like for (y)'s class by B, b, and d_1 respectively. Let the nth year be divided into m equal parts, m being any positive integer; then, confining our attention to one of the parts, say the tth in order, the deaths in this part will be, amongst (x)'s class $d \div m$, and amongst (y)'s $d_1 \div m$.

Hence the probability that (x) will die in the tth part is

$$\frac{1}{\mathrm{A}} \cdot \frac{d}{m}; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and that (y) will be alive at the end of it, the probability is,

$$\frac{1}{\mathrm{B}}\left(b - \frac{td_1}{m}\right), \text{ or } \frac{1}{m\mathrm{B}}(mb - td_1) \quad . \; . \; . \; . \quad (2).$$

Hence also, the probability of the concurrence of two independent events being the product of their individual probabilities, by multiplying together (1) and (2), we get

$$\frac{d}{m^2\mathrm{AB}}(mb - td_1) \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

for the probability that (x) will die in the tth part of the nth year and (y) be alive at the end of the same part.

Now this concurrence may happen in any one of the m parts into which the nth year is divided. Hence if in (3) we make $t = 1, 2, 3, \ldots m$, successively, and add the results, the sum will be the probability that (x) will die in, and (y) be alive at the end of, some one or other of the m parts; that is, that (x) will die in the nth year, and (y) survive him by a portion of time not exceeding one mth part of a year.

Replacing in the meantime $d \div m^2\mathrm{AB}$ by Q, we have,

$$\begin{array}{ll}
\text{for } t=1, & \mathrm{Q}(mb - d_1); \\
t=2, & \mathrm{Q}(mb - 2d_1); \\
t=3, & \mathrm{Q}(mb - 3d_1); \\
\vdots & \vdots \quad \vdots \\
t=m, & \mathrm{Q}(mb - md_1);
\end{array}$$

and summing, we get, on restoring the value of Q, since $1+2+3+\ldots+m=\frac{1}{2}m(m+1)$,

$$\frac{d}{m^2\mathrm{AB}}\left(m^2b-\frac{m(m+1)}{2}d_1\right), \text{ or } \frac{d}{\mathrm{AB}}\left(b-\frac{m+1}{2m}d_1\right), \quad . \quad . \quad . \quad (4)$$

for the probability in respect of the nth year, as above.

We can now, by giving to m in (4) the proper value, assign the probability that (x) will die in the nth year and be survived by (y) by a portion of time not exceeding any that we choose to name. Thus, for a period of survivorship not exceeding a month, making $m=12$, and observing that the coefficient of d_1 may be written $\frac{1}{2}+\frac{1}{2m}$, we have

$$\frac{d}{\mathrm{AB}}\left[b-\left(\frac{1}{2}+\frac{1}{24}\right)d_1\right];$$

and in like manner, for a term not exceeding a day, we have, making $m=365$,

$$\frac{d}{\mathrm{AB}}\left[b-\left(\frac{1}{2}+\frac{1}{730}\right)d_1\right].$$

From these instances it appears that by increasing m we diminish the term of (y)'s survivorship, and so assimilate the event, whose probability is thus assigned, more and more nearly to that whose probability is sought. Carrying this process to its limit, $1\div 2m$ vanishes, and we get finally, for the probability that (x) will die in the nth year, and (y) be alive *at the instant* of his death,

$$\frac{d}{\mathrm{AB}}(b-\tfrac{1}{2}d_1).$$

And this becomes, on replacing the symbols by their equivalents,

$$\frac{(l_{x+n-1}-l_{x+n})\{l_{y+n-1}-\frac{1}{2}(l_{y+n-1}-l_{y+n})\}}{l_{x.y}}$$

$$=\frac{(l_{x+n-1}-l_{x+n})(l_{y+n-1}+l_{y+n})}{2l_{x.y}};$$

or,

$$=\tfrac{1}{2}(p_{x.n-1}-p_{x.n})(p_{y.n-1}+p_{y.n}).$$

Multiplying these by v^n and summing with respect to n, we obtain the usual expressions for the present value of the assurance, which it is unnecessary to reproduce.

17. Do you consider the formula usually given for the foregoing benefit to be absolutely or only approximately true? If the latter, state the objection to it.

Ans. It has been already intimated that the expression here referred to, (which is the one just deduced,) is only approximately true. The number of survivors, at the expiry of the fraction $t\div m$ of the nth year (t being less than m) of l_{x+n-1} who enter on that year, is *truly* denoted by

$$l_{x+n-1}+\frac{t}{m}\Delta l_{x+n-1}+\frac{t(t-m)}{1.2m^2}\Delta^2 l_{x+n-1}+\ .\ .\ .\ .$$

the series being continued till the terms become insignificant.

In the foregoing investigation, in assuming that the deaths of the year are uniformly distributed over it, we in effect make use of the first two terms of the above development to denote the number living at the end of the time $t \div m$, neglecting all that follow. And hence the error in the resulting formula. It is not on account of any analytical difficulty in dealing with the complete development that we content ourselves with using the first two terms only: the reasons are, first, that the employment of any terms beyond the first two lands us in an expression far too complex to admit of being commodiously used; and, secondly, that the error committed by neglecting terms beyond the second is too small to be of any practical importance.

See *Assurance Magazine*, vol. i., pp. 137 to 151.

There still exists in the minds of a good many a lingering belief, notwithstanding the various demonstrations that have been given, that the expression under consideration is a comparatively rude approximation to the truth, and subject to at least one other source of error than that above referred to. We trust it will not be considered presumptuous in us to say, after the very explicit demonstration we have given, that there exists no reason for attributing to the expression deduced any other inaccuracy than that which arises from the assumption made as to the distribution of the deaths.

18. Find the annual premium for an assurance on the life of (x), deferred n years, and contingent upon (y) surviving that term. The premium to be payable for the term, provided that both (x) and (y) live so long.

Ans. Here the benefit term is

$$\frac{M_{x+n}}{D_x} \cdot \frac{l_{y+n}}{l_y} = \frac{M_{x+n} l_{y+n}}{D_{x.y}}, \text{ if } x > y;$$

and the payment term is

$$\frac{\varpi N_{x-1.y-1|n}}{D_{x.y}}.$$

Therefore, equating,

$$\varpi N_{x-1.y-1|n} = M_{x+n} l_{y+n},$$

whence,

$$\varpi = \frac{M_{x+n} l_{y+n}}{N_{x-1.y-1|n}}.$$

If $x < y$, a not very likely case to occur in practice, we should find,

$$\varpi = \frac{M_{x+n} l_{y+n} v^{y-x}}{N_{x-1.y-1|n}}.$$

19. Investigate a formula for the annual premium for an assurance of £1 payable in the event of a life aged x attaining the age $x+n$, with the provision that the premiums paid are to be returned (without interest) in the event of death happening before the attainment of the specified age. Allow for a loading of 10 per cent upon the premium.

Ans. The specialty in this question is as to the manner of applying the loading. This is a point which is not touched upon in any of the text-books, the reason for the omission no doubt being that in nearly all the cases that arise it suffices to apply the loading to the net premium. This is however obviously not sufficient when, as in the present case, there is to be a return of premium, since every increase of the premium involves also an increase of the return. In such cases therefore a different mode of treatment becomes necessary.

For generality, let the sum assured be A, let p be the loading per £, and let the required premium be ϖ.

The benefit consists of first, an endowment of A, receivable in n years; secondly, an assurance of A for n years; and thirdly, an increasing assurance of ϖ, 2ϖ, 3ϖ, &c. also for n years. These give for the value of the benefit (omitting the denominator, D_x)

$$A(D_{x+n}+M_{x|n})+\varpi R_{x|n};$$

and the value of the premium (subject to the like omission) is,

$$\varpi N_{x-1|n}.$$

If we were to form with these the usual equation, it is obvious that the resulting value of ϖ would be the *net* premium. But, by condition, the present value of the premium is to be $1+p$ times that of the benefit. Hence our equation is,

$$\varpi N_{x-1|n}=(1+p)\{A(D_{x+n}+M_{x|n})+\varpi R_{x|n}\};$$

and solution gives

$$\varpi=\frac{(1+p)A(D_{x+n}+M_{x|n})}{N_{x-1|n}-(1+p)R_{x|n}}.$$

This expression differs from that which we should obtain by applying the loading to the net premium, in having the *negative* term in the denominator multiplied by $1+p$. The value of ϖ is obviously thereby increased.

20. A person aged x is assured for m pounds, at an annual premium ϖ, payable until death. He desires to convert his assurance into one of the same amount, payable in n years or on previous death. Required the future premium ϖ', payable until the risk is determined.

Benefit Terms.

Ans. (x) receives, first, an assurance of m, payable at $x+n$ or death . . . . $m(D_{x+n}+M_{x|n})$;

and, secondly, remission of the premium ϖ, now payable . . . . ϖN_{x-1}.

Payment Terms.

He gives up his present assurance . . . . mM_x;

and he is to pay a premium ϖ', for n years . . . . $\varpi' N_{x-1|n}$.

Hence, equating the sums of the Benefit and the Payment Terms,

$$\varpi' N_{x-1|n}+mM_x=m(D_{x+n}+M_{x|n})+\varpi N_{x-1}$$

$$\therefore \quad \varpi'=\frac{m(D_{x+n}-M_{x+n})+\varpi N_{x-1}}{N_{x-1|n}}.$$

If the Office charge a commission on the transaction it will be sufficient in this case to apply the loading to the net premium ϖ', determined as above. See *Assurance Magazine*, vol. i., p. 98.

25. Will a loading of 25 per cent upon a net premium provide for an additional mortality also of 25 per cent? Give your reasons.

Ans. It is not easy to assign analytically the relation that subsists between a variation in the annual rate of mortality and the corresponding variation in either the single or the annual premium for assurance; but it is easy enough to show that there is not between them the simple relation that seems implied in the question.

First. The subsisting relation, whatever it is, must involve the rate of interest. The increase in the value of the liability on subsisting policies, consequent on an increase in the mortality, arises not from there being more claims to pay, but from the claims provided for by the premiums arising sooner than was anticipated. It will vary therefore with the rate of interest used in the calculation, and hence the implied relation, which ignores the rate of interest, cannot be that which actually subsists.

Secondly. Put the matter to the test by trying an extreme case. The mortality at age 94 (Carlisle) is ·25. If this be increased by 300 per cent, the value of an assurance on (94) will be $A_{94}=v=\cdot 9709$, at 3 per cent. Now subject to the normal mortality, and at 3 per cent interest, we have $A_{94}=\cdot 8912$ and $\varpi_{94}=\cdot 2386$, which become, when increased in the same ratio as the mortality, 3·5648 and ·9544, respectively. Hence as by the implied relation these ought to be each equal to ·9709, it inevitably follows that this relation is not that which actually subsists.

An addition of p per cent to a net premium will (leaving expenses, &c. out of view) exactly provide for a bonus of p per cent on the sum assured. But this is by no means the same thing as providing for an increase of p per cent in the annual mortality. In the one case the claims arise as anticipated, and their increased amounts will be duly met by the increased premiums; while in the other the claims arise sooner than was anticipated, and fewer payments of premiums are received.

ERRATA.

Mr. W. Sutton points out the following *errata* in Mr. Sprague's paper *On the Value of Apportionable Annuities.*

Vol. xiv., p. 38, line 4 from bottom,

$$\text{For } \tfrac{1}{4}+\frac{1}{2n}+\frac{1}{6n^2}-\frac{1}{30n^3} \text{ read } \tfrac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}.$$

Vol. xiv., p. 42, line 8,

$$\text{Add the term} \qquad +\frac{1}{12}\frac{D''_k}{D_k}\frac{\delta\tau^2}{2}\left(\frac{1}{m}-\tau\right)^2.$$